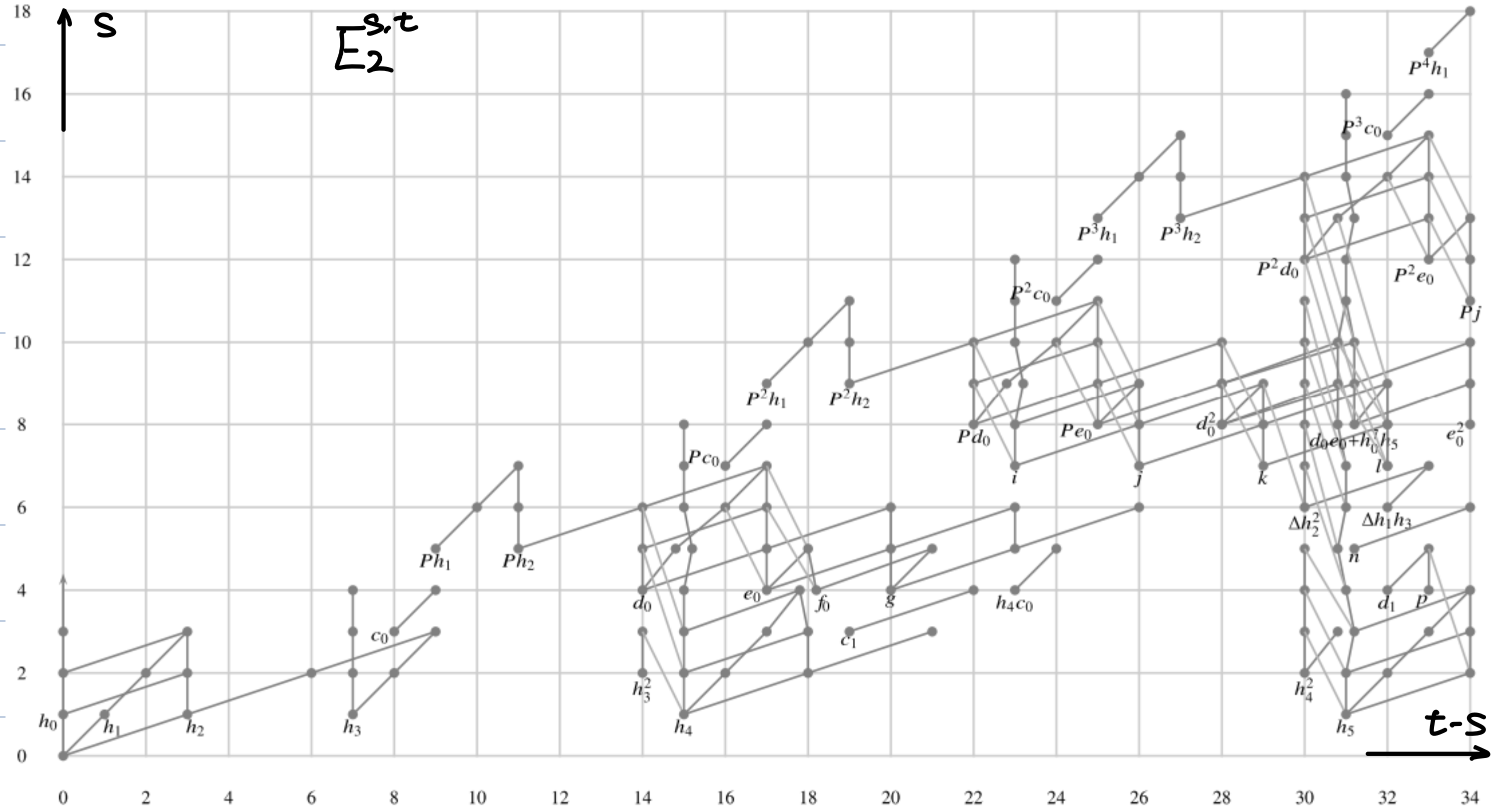
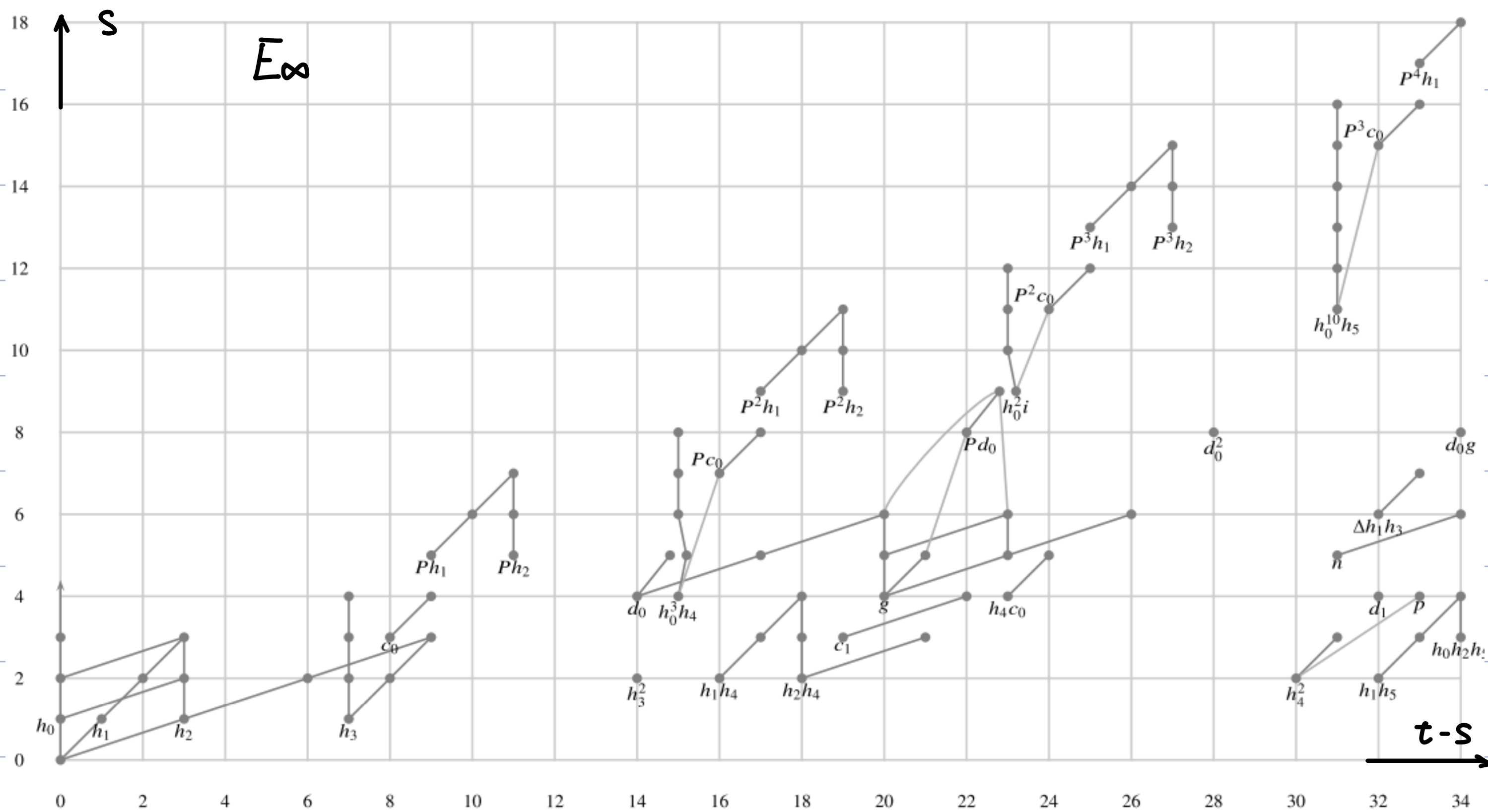


s
$E_2^{s,t}$
t-s
h_0
h_1
h_2
h_3
c_0
Ph_1
Ph_2
h_3^2
d_0
h_4
Pc_0
e_0
f_0
c_1
g
P^2h_1
P^2h_2
h_4c_0
Pd_0
i
P^2c_0
Pe_0
j
P^3h_1
P^3h_2
d_0^2
k
h_4^2
Δh_2^2
P^2d_0
$d_0e_0+h_0^7h_5$
h_5
n
P^3c_0
l
Δh_1h_3
d_1
p
P^2e_0
P^4h_1
Pj
e_0^2



s
E_∞
t-s
h_0
h_1
h_2
h_3
c_0
Ph_1
Ph_2
h_3^2
d_0
$h_0^3h_4$
Pc_0
h_1h_4
h_2h_4
c_1
g
P^2h_1
P^2h_2
h_4c_0
Pd_0
h_0^2i
P^2c_0
P^3h_1
P^3h_2
d_0^2
h_4^2
$h_0^{10}h_5$
n
P^3c_0
h_1h_5
d_1
Δh_1h_3
p
P^4h_1
d_0g
$h_0h_2h_5$

I. Collection of tools

E = Adams type spectra

$$\begin{tikzcd}
 & \mathrm{Sp} \arrow[d, "\nu"] \arrow[dl, bend right, "\mathrm{Id}"'] \arrow[ddrr, bend left, "E_*(-)"] & & \\
 & \mathrm{Syn}_E \arrow[dl, "(-)[\tau^{-1}]"'] \arrow[dr, "C\tau\otimes -"] & & \\
\mathrm{Sp} & & \mathrm{Mod}_{C\tau} \arrow[r, hookrightarrow] & \mathrm{Stable}_{E_*E}
\end{tikzcd}$$

① $\nu : \mathrm{Sp} \longrightarrow \mathrm{Shv}_\Sigma(\mathrm{Sp}_E^{fp}) \xrightarrow{\Sigma_+^\infty} \mathrm{Syn}_E$

$X \longmapsto \Sigma_+^\infty \mathrm{Map}_{\mathrm{Sp}_E^{fp}}(-, X)$

satisfies : 1) ν preserved filtered colim.

2) ν fully faithful & additive

3) $X \to Y \to Z$ cofib seq.

$\nu X \to \nu Y \to \nu Z$ cofib seq

$\Longleftrightarrow E_*X \to E_*Y \to E_*Z$ cof

4) ν strict sym. mon.

② $\tau : \nu S^{-1} \longrightarrow \Omega(\nu S^0)$.

Define $S^{t,w} = \Sigma^{t-w} \nu S^w$

Can show $\tau : S^{0,-1} \longrightarrow S^{0,0}$

$C\tau = \mathrm{cof}\, \tau =: S/\tau$.

③ $X \in \mathrm{Syn}_E$. X is said τ-invertible, if

$$\tau : \underset{\substack{\| \\ S^{0,-1}\wedge X}}{\Sigma^{0,-1}X} \xrightarrow{\simeq} X$$

FACT $\quad Sp \xrightarrow{\simeq} Syn(\tau^{-1})$ = all τ-invertible ...

Denote τ^{-1} = localization functor

FACT $\quad \tau^{-1}$ commute colims.

$\quad -\otimes C\tau \quad$ " $\quad$ "

Lem 1 $\quad X, Y \in Sp$.

$$[\nu Y, C\tau \otimes \nu X]_{t,w} \simeq \mathrm{Ext}^{w-t,w}_{E_*E}(E_*Y, E_*X).$$

Lem 2 $\quad X, Y \in Sp$. $\exists$ long exact seq.

$$\cdots \to [\nu Y, \nu X]_{t,w+1} \xrightarrow{\tau} [\nu X, \nu Y]_{t,w}$$
$$\to \mathrm{Ext}^{w-t,w}_{E_*E}(E_*Y, E_*X) \to [\nu Y, \nu X]_{t-1,w+1}$$
$$\to \cdots$$

Lem 3 $\quad X, Y \in Sp$. then $t-w \geq 0$

$$[\nu Y, \nu X]_{t,w} \xrightarrow[\tau^{-1}]{\simeq} [X, Y]_t.$$

In particular, $\quad \pi_{*,\leq 0}\, \nu X = \pi_* X[\tau]$

Cor 4 $\quad M \in Mod_E$. then $\pi_{*,*}\nu M \simeq \pi_* M \otimes_{\mathbb{Z}} \mathbb{Z}[\tau]$

$$\simeq \pi_* M[\tau].$$

If $M \simeq E \otimes X$, then

$$\nu E_{*,*}(\nu X) = \pi_{*,*}\, \nu(E \otimes X)$$
$$\simeq E_* X[\tau]$$

$\rightsquigarrow$ $(\nu E_{*,*} ,\ \nu E_{*,*}\nu E)$ (flat) Hopf algebroid.

Cor 5 $E = H\mathbb{F}_2$, $Y = \mathbb{S}$, $\nu Y = S^{0,0}$

$E_*E = \mathcal{A}$ Steenrod alg. $\mathcal{A}^* =$ dual Steenrod alg.

$\pi_{t-s,s}(\nu X \otimes C\tau) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}^*}(\mathbb{F}_2, \mathbb{F}_{2*}X)$.

Lem 6 $f: X \longrightarrow Y \in Sp$.

$$\begin{tikzcd} \nu(X) \arrow[r, "\nu(f)"] \arrow[dr, dashed, "\tilde{f}"'] & \nu(Y) \\ & \Sigma^{0,-k}\nu(Y) \arrow[u, "\tau^k"'] \end{tikzcd}$$

f of Adams filtration k.

[Burklund - Hahn - Senger Lem. 9.15]

Two tools :

1. Synthetic Adams SS.

Classically, E - Adams SS can be given by tower :

$$\begin{tikzcd} X = X_0 \arrow[d, "i_0"] & X_1 \arrow[l, "f_1"'] \arrow[d, "i_1"] & X_2 \arrow[l, "f_2"'] \arrow[d, "i_2"] & X_3 \arrow[l, "f_3"'] \arrow[d, "i_3"] & \cdots \arrow[l] \\ E\otimes X_0 & E\otimes X_1 & E\otimes X_2 & E\otimes X_3 & \end{tikzcd}$$

Corresponding νE - Adams tower :

$$\begin{tikzcd}
& \nu X_1 \arrow[d, "\nu f_1"'] \arrow[dr, "\tau"] & \Sigma^{0,1}\nu X_2 \arrow[d, "\nu f_2"] \arrow[dr, "\tau"] & \Sigma^{0,2}\nu X_3 \arrow[d, "\nu f_2"] \arrow[dr, "\tau"] & \Sigma^{0,3}\nu X_4 \arrow[d, "\nu f_3"] & \cdots \\
\nu X = & \nu X_0 \arrow[d, "i_0"] & \Sigma^{0,1}\nu X_1 \arrow[l, "\tilde{f}_1"'] \arrow[d, "i_1"] & \Sigma^{0,2}\nu X_2 \arrow[l, "\tilde{f}_2"'] \arrow[d, "i_2"] & \Sigma^{0,3}\nu X_3 \arrow[l, "\tilde{f}_3"'] \arrow[d, "i_3"] & \cdots \arrow[l] \\
& \nu E \otimes \nu X_0 & \nu E \otimes \Sigma^{0,1}\nu X_1 & \nu E \otimes \Sigma^{0,2}\nu X_2 & \nu E \otimes \Sigma^{0,3}\nu X_3 &
\end{tikzcd}$$

$\rightsquigarrow$

$$E_1^{s,t,w} = \pi_{t,t+w}(\nu E \otimes \Sigma^{0,s}\nu X_s) \Rightarrow \pi_{t,t+w}(\nu X).$$

$$d_r : E_r^{s,t,w} \longrightarrow E_r^{s+r,t-1,w+1}$$

$$|d_r| = (r, -1, 1), \qquad |\tau| = (0, 0, -1).$$

(s, t, s)

Thm 1) $E_1^{s,t,w} = E_1^{s,t} \otimes \mathbb{Z}[\tau]$

2) $E_2^{s,t,w} = E_2^{s,t} \otimes \mathbb{Z}[\tau]$

3) d_r. ASS $(x) = y \iff d_r$. SASS $(x) = \tau^{r-1}y$.

all diff. arise in this way.

[B-H-S Appendix A.1.]

Cor $r \geq 2$. $E_r^{s,t,w} = 0$. $w > s$

$E_r^{s,t,w} = E_r^{s,t}$ for $w \leq s-r+1$ and $w \leq 0$.

2. τ-Bockstein SS

Let $X \in \mathrm{Fil}\,\mathrm{Sp} = \mathrm{Fun}((\mathbb{Z}, \leq)^{op}, \mathrm{Sp})$.

e.g. $\cdots \longrightarrow X_3 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow \cdots$

Define "τ-Bockstein filtration". $F_\tau^* X$, to be

$$\begin{tikzcd} \cdots \arrow[r, "\tau"] & \Sigma^{0,-2}X \arrow[r, "\tau"] \arrow[d, dash] & \Sigma^{0,-1}X \arrow[r, "\tau"] \arrow[d, dash] & X = X \cdots \\ \cdots & gr^2 F_\tau X & gr^1 F_\tau X & \end{tikzcd}$$

$\rightsquigarrow$ exact couple $(w \geq 0)$

$$A^{s,t,w} = \pi_{s,t}(F_\tau^w X) \simeq \pi_{s,t}(\Sigma^{0,-w} X)$$
$$\simeq \pi_{s,t+w} X$$

$$E^{s,t,w} = \pi_{s,t}(gr^w F_\tau^* X) = \pi_{s,t+w}(X/\tau)$$

$\rightsquigarrow$ get a SS.

$$d_r : E_r^{s,t,w} \longrightarrow E_r^{s-1,t,w+r}$$

$$|d_r| = (-1, 0, r).$$

$$E_1^{s,t,w} \Rightarrow \pi_{s,t} X$$

What's more, can get an "τ-adic filtration on X":

$$F^w \pi_{s,t} X = \operatorname{im}(\tau^w : \pi_{s,t+w} X \longrightarrow \pi_{s,t} X).$$

Define $\bar{\tau}$, s.t. $|\bar{\tau}| = (0, -1, 1)$.

Goal: $$E_1^{s,t,w} \cong \underbrace{\pi_{s,t}(X/\tau)}_{(s,t,0)} \otimes \underbrace{\mathbb{Z}[\bar{\tau}]}_{\text{last index}}.$$

Can regrade $a \in E_1^{s,t,w}$ by "$\bar{\tau}^w \cdot a$" where a can be regarded as an elet of $\pi_{s,t}(X/\tau)$.

Note you really can do this!

Thm $X \in FilSp$. $\{E_r^{*,*}\}$ underlying SS for X

$\{E_r^{*,*,*}\}$ τ-Bockstein SS. Then

1) $\exists$ nat iso of $\mathbb{Z}[\bar{\tau}]$-alg :

$$E_1^{*,*,*} = E_1^{*,*} \otimes \mathbb{Z}[\bar{\tau}].$$

$$= \pi_{*,*}(X/\tau) \otimes \mathbb{Z}[\bar{\tau}].$$

2) $\forall\ r \geq 1$. $\exists$ surjective map

$$E_r^{s,t,w} \xrightarrow{F} E_r^{s,w}$$

iso for $w \geq r-1$. $\forall\ x \in E_r^{s,t,w}$. $y \in E_r^{s-1,t+r,w}$

then $\exists$ $d_r x = \bar{\tau}^r \cdot y$.

$\iff$ $d_r([x]) = [y]$ in $\{E_r^{*,*}\}$.

$[-]$ = image of F.

3) diffs are $\bar{\tau}$-linear : $\forall\ a, b \in E_r^{*,*,*}$ s.t.

$d_r a = b$.

then $\forall\ t \geq 0$, $d_r(\bar{\tau}^t a) = \bar{\tau}^t b$.

So $\mathbb{Z}[\bar{\tau}]$-mod on E_1 $\rightsquigarrow$ $\mathbb{Z}[\bar{\tau}]$-mod on E_r

$r \geq 1$.

4) $\forall\ x \in E_r^{s,t,w}$, $\exists\ y \in E_r^{s-1,t+r,w}$ (if $\exists$) s.t.

$d_r x = \bar{\tau}^r \cdot y$.

5) τ-Bockstein SS $\Rightarrow \pi_{*,*} X$ conditionally

iff $X \simeq \varprojlim_n X/\tau^n$.

Thm X E-nilpotent complete, i.e. E-Adams SS $\Rightarrow X$.
(e.g. $X \in Sp^{\omega}$, $E = H\mathbb{F}_p$, BP)

TFAE:

1) E-ASS converges strongly
2) νE-ASS " "
3) τ-Bockstein SS " "

If one holds, then
$$F^s \pi_{t,t+w}(\nu X) = F_\tau^{s-w} \pi_{t,t+w}(\nu X)$$
if $s = t$, $F_\tau^t \pi_{t,t+w}(\nu X) = \pi_{t,t+w}(\nu X)$

Cor $F^s \pi_t X = \mathrm{im}(\pi_{t,t+s} X \xrightarrow{\tau^{-1}} \pi_t X)$.
is the Adams filtration.

Def E-Adams SS $\overset{\text{strongly}}{\Longrightarrow} X$ if

1) $F^\bullet \pi_* X$ is complete & Hausdorff.

$\lim^1 \simeq 0$ $\bigcap_s F^s \pi_* X = 0$

2) $F^s \pi_{t-s} X / F^{s+1} \pi_{t-s} X \cong E_\infty^{s,t}$.

II. Computational Examples: $\pi_{s,t}(\nu \hat{\mathbb{S}}_2)$, $s \le 19$

In practice $s \le 13$.

Thm [B-H-S Thm 9.19]

X E-nilpotent complete. $X \in Sp$, and E-ASS converge strongly. Then:

1. $x \in F^s \pi_k X$ in E_2-page. TFAE: $(r \geq 2)$

① $d_2, d_3, \dots, d_r$ vanish on x

② x, regarded $\pi_{k,k+s}(C\tau \otimes \nu X)$, lifts to $\pi_{k,k+s}(C\tau^r \otimes \nu X)$

③ $x \rightsquigarrow$ sth. in $\pi_{k,k+s}(C\tau^r \otimes \nu X)$. image under τ-Bockstein:

$$C\tau^r \otimes \nu X \longrightarrow \Sigma^{1,-r} C\tau \otimes \nu X.$$

is equal to $-d_{r+1}(x)$.

2. (based on 1). Assume x is a permanent cycle. $\exists$ x lift (not nec. unique) along

$$\pi_{k,k+s}(\nu X) \longrightarrow \pi_{k,k+s}(C\tau \otimes \nu X).$$

For any such lift, $\tilde{x}$, the following are true:

① if x survive E_{r+1}-page. $\tau^{r-1}\tilde{x} \neq 0$.

② .. .. E_∞-page. then $\tilde{x}$ in $\pi_k X$ of E-Adams filtration s. and $\tilde{x}$ can be detected by x.

3. (based 2). $\exists$ $\tilde{x}$ s.t.

① if $x = d_{r+1}(y)$ for some y. then can choose $\tilde{x}$ s.t. $\tau^r \tilde{x} = 0$.

② if x survives, $\alpha \in \pi_k X$ detects x.

and choose $\tilde{x}$ s.t. $\tau^{-1}\tilde{x} = \alpha$.

Write $\tilde{\alpha} := \tilde{x}$.

4. (based on 3).

Fix any collection $\tilde{x}$ (lift x) s.t. x span the permanent cycles in top deg k. Then

$\pi_{k,*}(\nu X)$ viewed $\mathbb{Z}[\tau]$-mod

its τ-adic completion gen. by $\tilde{x}$ $= \pi_{k,*}(\nu X)$.

pf. [B-H-S Appendix A.1]

- Computation of $\pi_{k,*}\, \nu \mathbb{S}_2^{\wedge} =: \pi_{k,*}$.

1. $S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow S/\tau$.

Recall $\pi_{t-s,s}(C\tau \otimes \nu X) = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_{2*}, \mathbb{F}_{2*}X)$

Take $X = \mathbb{S}$, $t = s$, $s \in \mathbb{Z}$.

$\rightsquigarrow$ $\pi_{0,*}(S/\tau) = \mathrm{Ext}_{A_*}^{*,*}(\mathbb{F}_{2*}, \mathbb{F}_{2*})$.

$= \mathbb{F}_2[h_0]$.

2. $\pi_{0,*}$

$$\rightsquigarrow \begin{tikzcd} \pi_{0,*} \arrow[r, "\text{quotient}"] \arrow[d, hook] & \pi_{0,*} S/\tau = \mathbb{F}_2[h_0]. \\ \pi_{0,*}[\tau^{-1}] \cong \mathbb{Z}_2[\tau, \tau^{-1}] & \end{tikzcd}$$

since $\pi_{0,*}S$ τ-power torsion

free.

<u>FACT</u> No differentials in $H\mathbb{F}_2$ - ASS for $\mathbb{S}_2^\wedge$ in top deg ≤ 13.

So $\pi_{0,*}$ can be regarded $\mathbb{Z}_2[\tau]$-submodule of $\mathbb{Z}_2[\tau^{\pm}]$ (note $\pi_{k,\leq 0} = \pi_k$)

tells us :

$$\begin{pmatrix} \text{something in } \pi_{0,*} \\ \text{is } \tau\text{-divisible} \end{pmatrix} \iff \begin{pmatrix} \cdots \text{ maps to} \\ 0 \text{ in } \pi_{0,*} S/\tau \end{pmatrix}$$

Note
$$\tau \longmapsto 0$$
$$1 \longmapsto 1$$
$$2 \longmapsto 0 \rightsquigarrow 2 \text{ is divisible by } \tau.$$
$$\rightsquigarrow \tilde{2} := 2/\tau \in \pi_{0,*}$$
$$2^n \longmapsto 0 \rightsquigarrow \tilde{2}^n = (2/\tau)^n$$

$$\rightsquigarrow \pi_{0,*} = \mathbb{Z}_2[\tau, \tilde{2}] / \tau \cdot \tilde{2} = 2.$$

3. $\pi_{1,*}$

τ-power torsion free ! h_1 survives. $\Rightarrow$ h_1 lifts to $\pi_{1,2}$ lift is unique.

$$h_1 \leftrightsquigarrow \tilde{\eta} \in \pi_{1,2}$$

$$\tilde{2} \cdot \tilde{\eta} = 0 \quad b/c \quad \tilde{2} \cdot \tilde{\eta} \in \underbrace{\pi_{1,3} = 0}$$

$\mathrm{Ext}^{2,s} = 0, \ s \geq 2$?

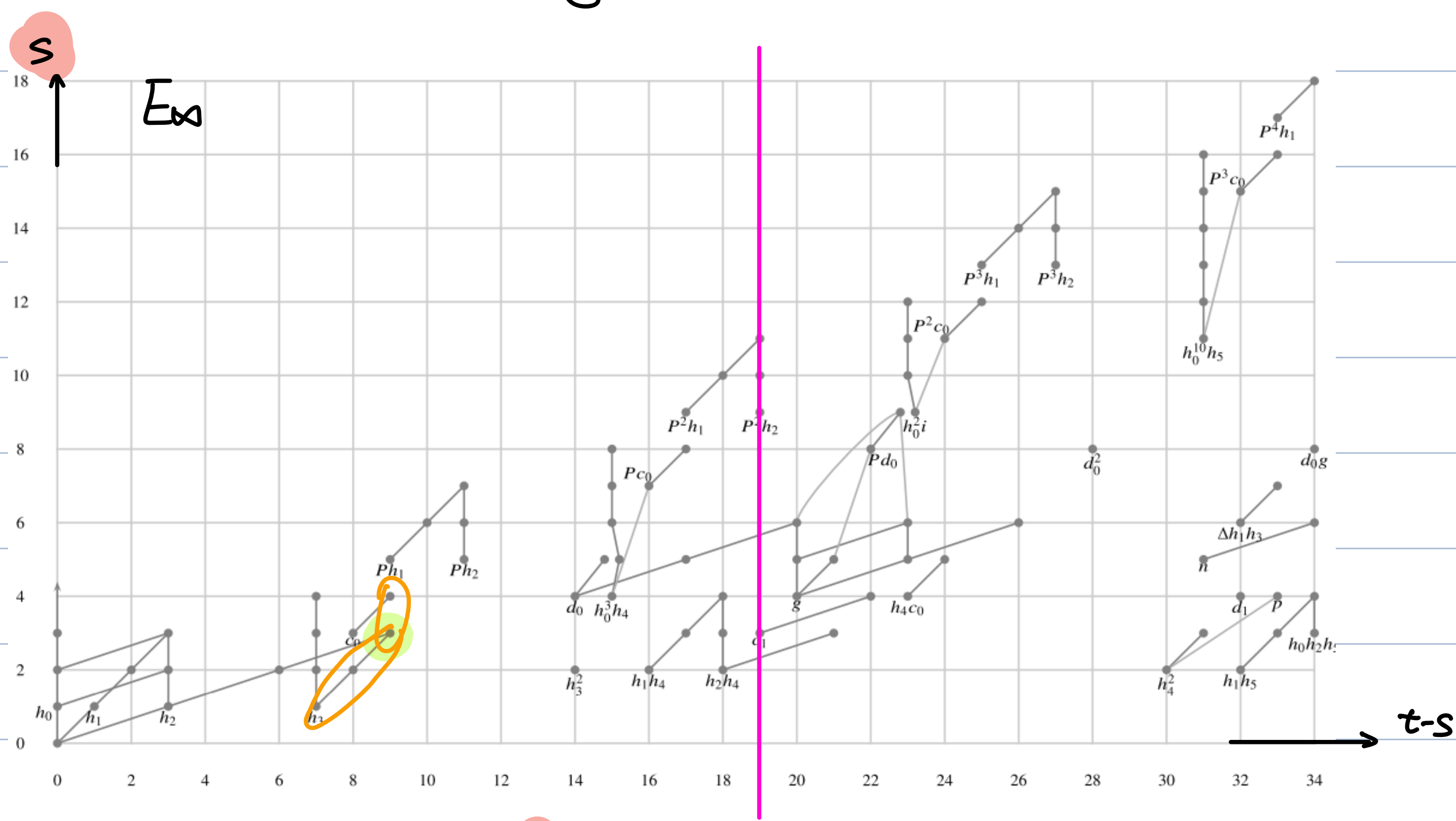
$\Rightarrow \quad \pi_{1,*} = \mathbb{Z}_2[\tau, \tilde{2}]\langle\tilde{\eta}\rangle / \tilde{2}\tilde{\eta} = 0.$

Can proceed.

<u>Slogan</u> Can construct $\pi_{k,*}$ without knowledge in π_k. Only input $\mathrm{Ext}^{s,t}_{A^*}(\mathbb{F}_2, \mathbb{F}_2)$.

4. $\pi_{\le 19, *}$

By Thm 9.19. $\forall$ generator (if survive to E_∞), can be lifted to a generator of $\pi_{\le 19, *}$.



π_k		$\pi_{k,k+s}$
$h_0 \rightsquigarrow \tilde{2}$		$\pi_{0,1}$
$h_1 \rightsquigarrow \tilde{\eta}$		$\pi_{1,2}$
$h_2 \rightsquigarrow \tilde{\nu}$		$\pi_{3,4}$
$h_3 \rightsquigarrow \tilde{\sigma}$		$\pi_{7,8}$

$c_0 \rightsquigarrow \tilde{\varepsilon}$	$\pi_{8,11}$
$Ph_1 \rightsquigarrow \widetilde{Ph_1}$	$\pi_{9,14}$
$Ph_2 \rightsquigarrow \widetilde{Ph_2}$	$\pi_{11,16}$
$d_0 \rightsquigarrow \tilde{\kappa}$	$\pi_{14,18}$

$h_0^3 h_4$	$\rightsquigarrow$	$\tilde{\rho}$	$\pi_{15,19}$	$h_2 h_4$	$\rightsquigarrow$	$\tilde{\nu}^*$	$\pi_{18,20}$
Pc_0	$\rightsquigarrow$	$\widetilde{Pc_0}$	$\pi_{16,23}$	c_1	$\rightsquigarrow$	$\tilde{c}_1$	$\pi_{19,22}$
$h_1 h_4$	$\rightsquigarrow$	$\tilde{\eta}^*$	$\pi_{16,18}$	$P^2 h_2$	$\rightsquigarrow$	$\widetilde{P^2 h_2}$	$\pi_{19,28}$
$P^2 h_1$	$\rightsquigarrow$	$\widetilde{P^2 h_1}$	$\pi_{17,26}$			τ	$\pi_{0,-1}$

Rk Hidden extensions in E_∞-page is reflected by the appearance of τ. Moreover, they can be detected directly by the second index.

e.g.
$$\tilde{\nu}^3 = \tilde{\eta}^2 \tilde{\sigma} + \tau \tilde{\eta} \tilde{\varepsilon}$$
$$\tilde{\eta} \tilde{\rho} = \tau^2 \widetilde{Pc_0}.$$

e.g. $\tilde{\nu}\, \widetilde{Ph_2} = \tilde{\varepsilon}^2 \tilde{\kappa}$

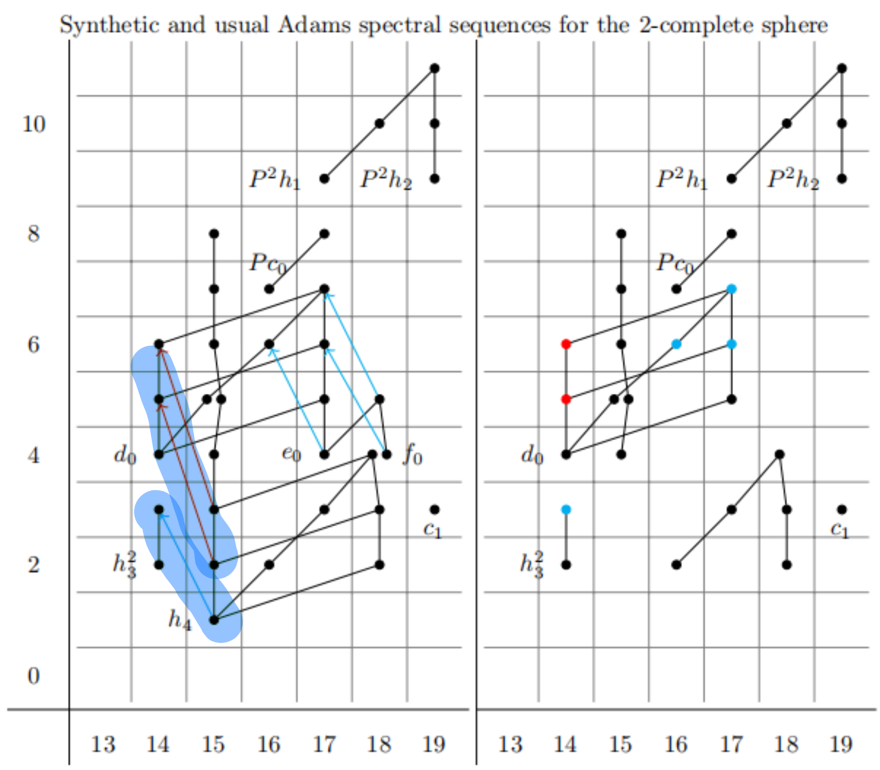


FIGURE 4. Left: Adams spectral sequence for the sphere, with differentials color-coded by length. Right: E_∞-page of the synthetic Adams spectral sequence for $\nu_{\mathrm{HF}_2}\mathbb{S}_2^\wedge$. Black dots indicate a copy of $\mathbb{F}_2[\tau]$, red dots indicate a copy of $\mathbb{F}_2[\tau]/\tau^2$ and blue dots indicate a copy of $\mathbb{F}_2[\tau]/\tau$.

Relations like : $\widetilde{\eta}^3 = \widetilde{\Sigma}^2 \widetilde{\upsilon}$

$\widetilde{\Sigma}^2 \widetilde{\upsilon^*} = \widetilde{\eta^*}\, \widetilde{\eta}^2$

...

can be directly read off from classical Adam SS.

Relations like $0 = 2\widetilde{\sigma}^2$: Koszul sign.

$$S^{t+t', w+w'} \simeq S^{t,w} \otimes S^{t',w'} \simeq S^{t',w} \otimes S^{t,w} \simeq S^{t'+t, w'+w}$$

$\text{sign} = (-1)^{tt'}$

Relations like $2\widetilde{\upsilon}\widetilde{\kappa} = 0 = \tau \widetilde{\eta}^2 \widetilde{\kappa}$:

Note not τ-power torsion free!

$\pi_{14,17}$, $\pi_{14,18}$. $\pi_{14,19}$

$\pi_{14,20}$. $\pi_{16,22}$, $\pi_{17,23}$

$\pi_{17,24}$

$= 0$ $\sim$ τ^r-divisible for some r

$r > 1$.

by deg.

Relations like $\widetilde{\upsilon}\widetilde{Ph_2} = \widetilde{\Sigma}^2 \widetilde{\kappa}$ or

$\widetilde{\varepsilon}^2 = \widetilde{\eta}^2 \widetilde{\kappa} = \widetilde{\sigma}\, \widetilde{Ph_1} + \tau \widetilde{Pc_0}$:

Much harder!

Relation chart :

1) $\tilde{\eta}^3 = \tilde{2}^2 \tilde{\nu}$

2) $\tilde{\eta}\tilde{\rho} = \tau^2 \widetilde{Pc_0}$

3) $\tilde{\nu}^3 = \tilde{\eta}^2 \tilde{\sigma} + \tau \tilde{\eta} \tilde{\varepsilon}$

Hidden ext is reflected by the appearance of τ.

4) $\tilde{\eta}^2 \widetilde{Ph_1} = \tilde{2}^2 \widetilde{Ph_2}$

5) $\tilde{2}^2 \tilde{\nu}^* = \tilde{\eta}^* \tilde{\eta}^2$.

So hidden ext is detected by the second grading !!

6) $\tilde{\varepsilon}\, \widetilde{Ph_1} = \tilde{\eta}\, \widetilde{Pc_0}$

7) $\widetilde{Ph_1}^2 = \tilde{\eta}\, \widetilde{P^2h_1}$

8) $\tilde{\eta}^2 \widetilde{P^2h_1} = \tilde{2}^2 \widetilde{P^2h_1}$.

9)
$$\begin{aligned} 0 &= \tilde{2}\tilde{\eta} = \tilde{\eta}\tilde{\nu} = \tilde{2}\tilde{\nu}^2 = \tilde{2}^4\tilde{\sigma} = \tilde{\nu}\tilde{\sigma} \\ &= \tilde{\eta}\tilde{\sigma}^2 = \tilde{2}\tilde{\varepsilon} = \tilde{\eta}^2\tilde{\varepsilon} = \tilde{\nu}\tilde{\varepsilon} = \tilde{\sigma}\tilde{\varepsilon} \\ &= \tilde{2}\,\widetilde{Ph_1} = \tilde{\nu}\,\widetilde{Ph_1} = \tilde{\eta}\,\widetilde{Ph_2} = \tilde{\sigma}\,\widetilde{Ph_2} \\ &= \tilde{\varepsilon}\,\widetilde{Ph_2} = \tilde{2}^3\tilde{\kappa} = \tilde{2}^5\tilde{\rho} = \tilde{\nu}\tilde{\rho} = \tilde{2}\,\widetilde{Pc_0} \\ &= \tilde{\eta}^2\,\widetilde{Pc_0} = \tilde{\nu}\,\widetilde{Pc_0} = \tilde{2}\tilde{\eta}^* = \tilde{\nu}\tilde{\eta}^* \\ &= \tilde{2}\,\widetilde{P^2h_1} = \tilde{\eta}\tilde{\nu}^* = \tilde{2}\tilde{c}_1 . \end{aligned}$$

10) $\tau \tilde{2} = 2$

11) $0 = 2\tilde{\sigma}^2$

12) $0 = 2\tilde{\nu}\tilde{\kappa}$

13) $0 = \tau\tilde{\eta}^2\tilde{\kappa}$

14) $\tilde{\nu}\,\widetilde{Ph_2} = \tilde{2}^2\tilde{\kappa}$

15) $2\tilde{\kappa} = \tilde{2}^2\tilde{\sigma}^2$

16) $\tilde{\varepsilon}^2 = \tilde{\eta}^2\tilde{\kappa} = \tilde{\sigma}\,\widetilde{Ph_1} + \tau\widetilde{Pc_0}$.

For (11): Note that $\nu\mathbb{S}_2^\wedge$ has an E_∞-str. In that case, by Rk 4.10 in Piotr, $\pi_{*,*}\nu\mathbb{S}_2^\wedge$ forms a bigraded ring which is comm. in the sense that the Koszul sign rule applied in top deg (the first one).

i.e. sign of
$$S^{t+t', w+w'} \simeq S^{t,w} \otimes S^{t',w'}$$
$$\simeq S^{t',w'} \otimes S^{t,w}$$
$$\simeq S^{t'+t, w'+w}$$
is $(-1)^{tt'}$.

Since $\tilde{\sigma} \in \pi_{7,8}$, $\tilde{\sigma}\cdot\tilde{\sigma} = (-1)^{7\cdot 7}\tilde{\sigma}\tilde{\sigma}$
$$\Rightarrow 2\tilde{\sigma}^2 = 0.$$

For (12), (13), note that classically, $2\nu\kappa = \eta^2\kappa = 0$

So synthetically, they don't have to be 0, but can be τ-power torsion. But we've known the nonzero τ-power torsion options, $\tilde{2}\tilde{\nu}\tilde{\kappa} \in \pi_{17,23}$, $\tilde{\eta}^2\tilde{\kappa} \in \pi_{16,22}$.

Claim $\pi_{17,23}^{tor}$, $\pi_{16,22}^{tor}$ only contain simple τ-torsion.

$$\Rightarrow \begin{cases} \tau\tilde{2}\tilde{\nu}\tilde{\kappa} = 0 \\ \tau\tilde{\eta}^2\tilde{\kappa} = 0 \\ \tau\tilde{2} = 2 \end{cases} \Rightarrow \begin{cases} 2\tilde{\nu}\tilde{\kappa} = 0 \\ \tau\tilde{\eta}^2\tilde{\kappa} = 0 \end{cases}$$